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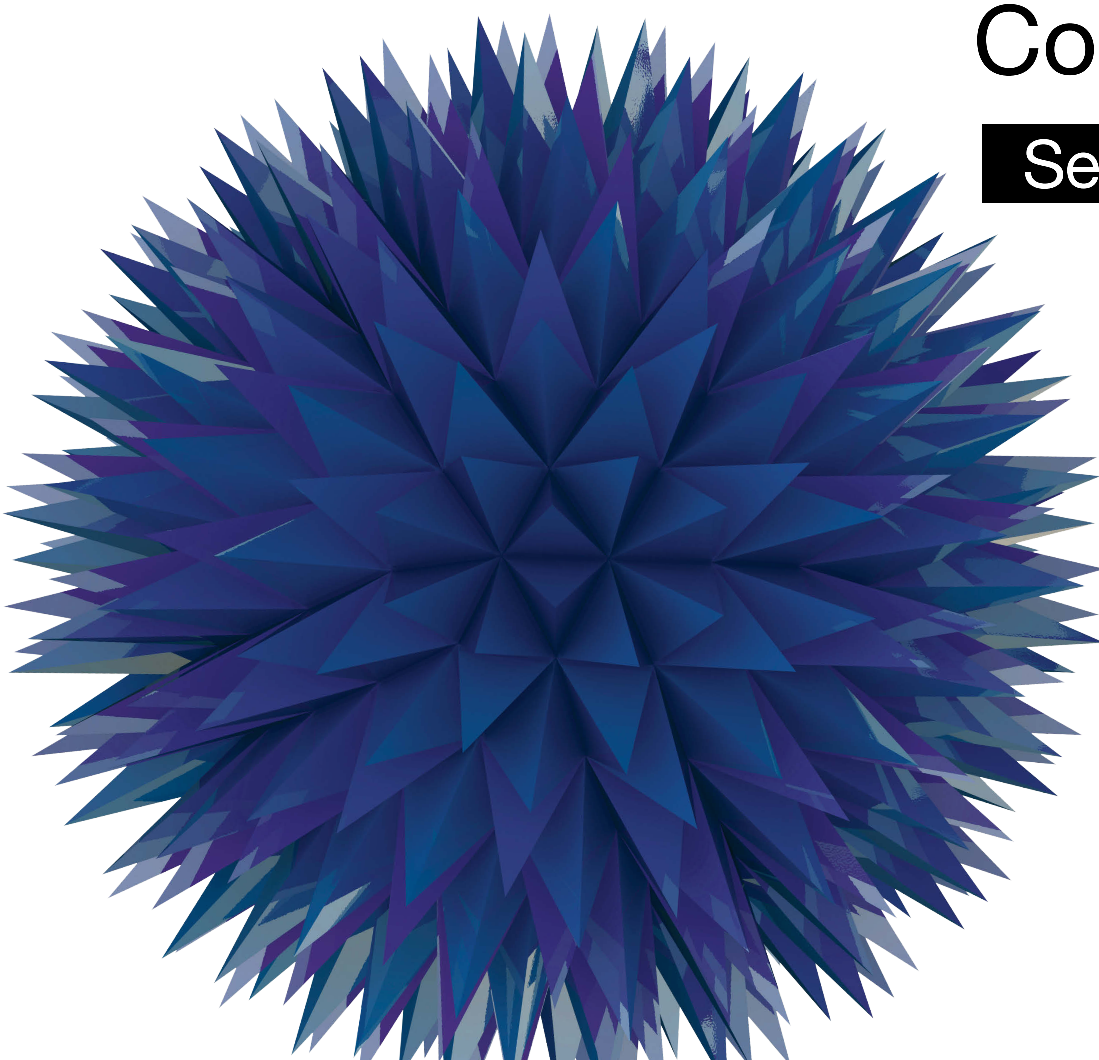
Sue Pemberton

Cambridge IGCSE® and O Level
Additional Mathematics

Coursebook

Second edition



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Contents

Chapter 11
Series

This section will show you how to:

- use the binomial theorem for expansion of $(a+b)^n$ for positive integral n
- use the general term $\binom{n}{r}a^{n-r}b^r$ for a binomial expansion
- recognise arithmetic and geometric progressions
- use the formula for the nth term and for the sum of the first n terms to solve problems involving arithmetic and geometric progressions
- use the condition for the convergence of a geometric progression, and the formula for the sum to infinity of a convergent geometric progression.

11.1 Pascal's triangle

The word 'binomial' means 'two terms'.

The word is used in algebra for expressions such as $x + 5$ and $2x - 3y$.

You should already know that $(a+b)^2 = (a+b)(a+b) = a^2 + 2ab + b^2$.

The expansion of $(a+b)^2$ can be used to expand $(a+b)^3$.

$$\begin{aligned}(a+b)^3 &= (a+b)(a+b)^2\\ &= (a+b)(a^2+2ab+b^2)\\ &= a^3 + 2a^2b + ab^2 + a^2b + 2ab^2 + b^3\\ &= a^3 + 3a^2b + 3ab^2 + b^3\end{aligned}$$

Similarly it can be shown that $(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$.

Writing the expansions of $(a+b)^n$ out in order:

$$\begin{aligned}(a+b)^1 &= 1a + 1b\\ (a+b)^2 &= 1a^2 + 2ab + 1b^2\\ (a+b)^3 &= 1a^3 + 3a^2b + 3ab^2 + 1b^3\\ (a+b)^4 &= 1a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + 1b^4\end{aligned}$$

If you look at the expansion of $(a+b)^4$, you should notice that the powers of a and b form a pattern.

- The first term is a^4 and then the power of a decreases by 1 whilst the power of b increases by 1 in each successive term.
- All of the terms have a total index of 4 (a^4, a^3b, a^2b^2, ab^3 and b^4).

There is a similar pattern in the other expansions.

The coefficients also form a pattern that is known as **Pascal's triangle**.

1 1

1 2 1

1 3 3 1

1 4 6 4 1

Note:

- Each row always starts and finishes with a 1.
- Each number is the sum of the two numbers in the row above it.

The next row would be:

1 5 10 10 5 1

This row can then be used to write down the expansion of $(a+b)^5$.

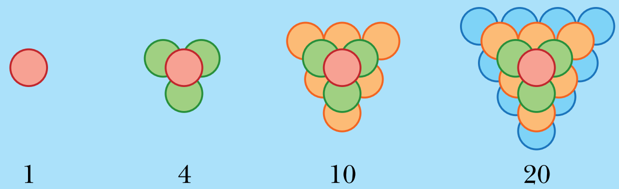
$$(a+b)^5 = 1a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + 1b^5$$

CLASS DISCUSSION

1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

There are many number patterns in Pascal's triangle.

For example, the numbers 1, 4, 10 and 20 have been highlighted.



These numbers are called tetrahedral numbers.

Which other number patterns can you find in Pascal's triangle?

What do you notice if you find the total of each row in Pascal's triangle?

WORKED EXAMPLE 1

Use Pascal's triangle to find the expansion of:

a $(2 + 5x)^3$ **b** $(2x - 3)^4$

Answers

a $(2 + 5x)^3$

The index = 3 so use the 3rd row in Pascal's triangle.

The 3rd row of Pascal's triangle is 1, 3, 3 and 1.

$(2 + 5x)^3 = 1(2)^3 + 3(2)^2(5x) + 3(2)(5x)^2 + 1(5x)^3$ Use the expansion of $(a + b)^3$.

$= 8 + 60x + 150x^2 + 125x^3$

b $(2x - 3)^4$

The index = 4 so use the 4th row in Pascal's triangle.

The 4th row of Pascal's triangle is 1, 4, 6, 4 and 1.

$(2x - 3)^4 = 1(2x)^4 + 4(2x)^3(-3) + 6(2x)^2(-3)^2$ Use the expansion of $(a + b)^4$.

$+ 4(2x)(-3)^3 + 1(-3)^4$

$= 16x^4 - 96x^3 + 216x^2 - 216x^3 + 81$

WORKED EXAMPLE 2

a Expand $(2 - x)^5$.

b Find the coefficient of x^3 in the expansion of $(1 + 3x)(2 - x)^5$.

Answers

a $(2 - x)^5$

The index = 5 so use the 5th row in Pascal's triangle.

The 5th row of Pascal's triangle is 1, 5, 10, 10, 5 and 1.

$$(2 - x)^5 = 1(2)^5 + 5(2)^4(-x) + 10(2)^3(-x)^2 + 10(2)^2(-x)^3 + 5(2)(-x)^4 + 1(-x)^5$$
$$= 32 - 80x + 80x^2 - 40x^3 + 10x^4 - x^5$$

b $(1 + 3x)(2 - x)^5 = (1 + 3x)(32 - 80x + 80x^2 - 40x^3 + 10x^4 - x^5)$

The term in x^3 comes from the products:

$$(1 + 3x)(32 - 80x + 80x^2 - 40x^3 + 10x^4 - x^5)$$

$1 \times (-40x^3) = -40x^3$ and $3x \times 80x^2 = 240x^3$

So the coefficient of x^3 is $-40 + 240 = 200$.

Exercise 11.1

1 Write down the 6th and 7th rows of Pascal's triangle.

2 Use Pascal's triangle to find the expansions of:

a $(1 + x)^3$ **b** $(1 - x)^4$ **c** $(p + q)^4$ **d** $(2 + x)^3$

e $(x + y)^5$ **f** $(y + 4)^3$ **g** $(a - b)^3$ **h** $(2x + y)^4$

i $(x - 2y)^3$ **j** $(3x - 4)^4$ **k** $\left(x + \dfrac{2}{x}\right)^3$ **l** $\left(x^2 - \dfrac{1}{2x^3}\right)^3$

3 Find the coefficient of x^3 in the expansions of:

a $(x + 4)^4$ **b** $(1 + x)^5$ **c** $(3 - x)^4$ **d** $(3 + 2x)^3$

e $(x - 2)^5$ **f** $(2x + 5)^4$ **g** $(4x - 3)^5$ **h** $\left(3 - \dfrac{1}{2}x\right)^4$

4 $(4 + x)^5 + (4 - x)^5 = A + Bx^2 + Cx^4$

Find the value of A, the value of B and the value of C.

5 Expand $(1 + 2x)(1 + 3x)^4$.

6 The coefficient of x in the expansion of $(2 + ax)^3$ is 96.

Find the value of the constant a.

7 **a** Expand $(3 + x)^4$.

b Use your answer to **part a** to express $\left(3 + \sqrt{5}\right)^4$ in the form $a + b\sqrt{5}$.

8 **a** Expand $(1+x)^5$.

b Use your answer to **part a** to express

i $(1+\sqrt{3})^5$ in the form $a+b\sqrt{3}$, **ii** $(1-\sqrt{3})^5$ in the form $c+d\sqrt{3}$.

c Use your answers to **part b** to simplify $(1+\sqrt{3})^5+(1-\sqrt{3})^5$.

9 **a** Expand $(2-x^2)^4$.

b Find the coefficient of x^6 in the expansion of $(1+3x^2)(2-x^2)^4$.

10 Find the coefficient of x in the expansion $\left(x-\frac{3}{x}\right)^5$.

11 Find the term independent of x in the expansion of $\left(x^2+\frac{1}{2x}\right)^3$.

CHALLENGE Q

12 **a** Find the first three terms, in ascending powers of y, in the expansion of $(2+y)^5$.

b By replacing y with $3x-4x^2$, find the coefficient of x^2 in the expansion of $(2+3x-4x^2)^5$.

CHALLENGE Q

13 The coefficient of x^3 in the expansion of $(3+ax)^5$ is 12 times the coefficient of x^2 in the expansion of $\left(1+\frac{ax}{2}\right)^4$. Find the value of a.

CHALLENGE Q

14 **a** Given that $\left(x^2+\frac{4}{x}\right)^3-\left(x^2-\frac{4}{x}\right)^3=ax^3+\frac{b}{x^3}$, find the value of a and the value of b.

b Hence, without using a calculator, find the exact value of

$$\left(2+\frac{4}{\sqrt{2}}\right)^3-\left(2-\frac{4}{\sqrt{2}}\right)^3.$$

CHALLENGE Q

15 Given that $y=x+\frac{1}{x}$, express:

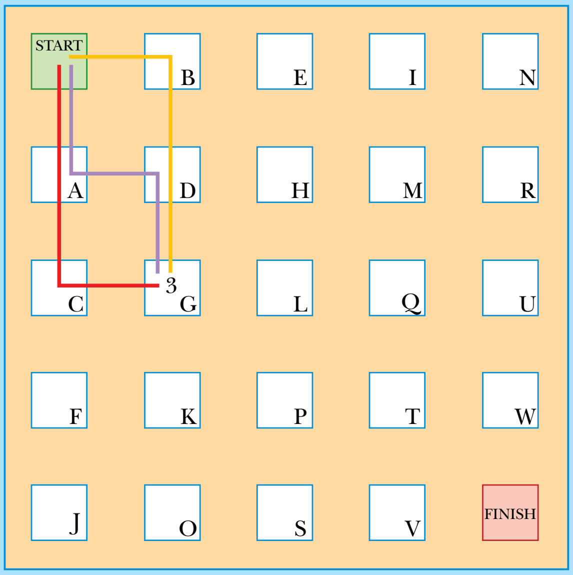
a $x^3+\frac{1}{x^3}$ in terms of y

b $x^5+\frac{1}{x^5}$ in terms of y.

CLASS DISCUSSION

The stepping stone game

The rules are that you can move East → or South ↓ from any stone.



The diagram shows there are 3 routes from the START stone to stone G.

1 Find the number of routes from the START stone to each of the following stones:

a **i** A **ii** B

b **i** C **ii** D **iii** E

c **i** F **ii** G **iii** H **iv** I

What do you notice about your answers to **parts a**, **b** and **c**?

2 There are 6 routes from the START to stone L.

How could you have calculated that there are 6 routes without drawing or visualising them?

3 What do you have to do to find the number of routes to any stone?

4 How many routes are there from the START stone to the FINISH stone?

In the class discussion you should have found that the number of routes from the START stone to stone Q is 10.

To move from START to Q you must move East (E) 3 and South (S) 2, in any order.

Hence the number of routes is the same as the number of different combinations of 3 E's and 2 S's.

The combinations are:

EEESS	EESES	EESSE	ESESE	ESEES
ESSEE	SSEEE	SESEE	SEESE	SEEES

So the number of routes is 10.

This is the same as ${}^5C_3$ (or ${}^5C_2$).

11.2 The binomial theorem

Pascal's triangle can be used to expand $(a + b)^n$ for any positive integer n, but if n is large it can take a long time. Combinations can be used to help expand binomial expressions more quickly.

Using a calculator:

$^5C_0 = 1 \quad ^5C_1 = 5 \quad ^5C_2 = 10 \quad ^5C_3 = 10 \quad ^5C_4 = 5 \quad ^5C_5 = 1$

These numbers are the same as the numbers in the 5th row of Pascal's triangle.

So the expansion of $(a + b)^5$ is:

$$(a + b)^5 = {}^5C_0\, a^5 + {}^5C_1\, a^4 b + {}^5C_2\, a^3 b^2 + {}^5C_3\, a^2 b^3 + {}^5C_4\, ab^4 + {}^5C_5\, b^5$$

This can be written more generally as:

$$(a + b)^n = {}^nC_0\, a^n + {}^nC_1\, a^{n-1} b + {}^nC_2\, a^{n-2} b^2 + {}^nC_3\, a^{n-3} b^3 + \ldots + {}^nC_r\, a^{n-r} b^r + \ldots + {}^nC_n\, b^n$$

But ${}^nC_0 = 1$ and ${}^nC_n$=1, so the formula simplifies to:

$$(a + b)^n = a^n + {}^nC_1\, a^{n-1} b + {}^nC_2\, a^{n-2} b^2 + {}^nC_3\, a^{n-3} b^3 + \ldots + {}^nC_r\, a^{n-r} b^r + \ldots + b^n$$

or

$$(a + b)^n = a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \binom{n}{3} a^{n-3} b^3 + \ldots + \binom{n}{r} a^{n-r} b^r + \ldots + b^n$$

The formulae above are known as the **binomial theorem**.

WORKED EXAMPLE 3

Use the binomial theorem to expand $(3 + 4x)^5$.

Answers

$(3 + 4x)^5 = 3^5 + {}^5C_1\, 3^4(4x) + {}^5C_2\, 3^3(4x)^2 + {}^5C_3\, 3^2(4x)^3 + {}^5C_4\, 3(4x)^4 + (4x)^5$

$\qquad = 243 + 1620x + 4320x^2 + 5760x^3 + 3840x^4 + 1024x^5$

WORKED EXAMPLE 4

Find the coefficient of x^{20} in the expansion of $(2 - x)^{25}$.

Answers

$(2 - x)^{25} = 2^{25} + {}^{25}C_1\, 2^{24}(-x) + {}^{25}C_2\, 2^{23}(-x)^2 + \ldots + {}^{25}C_{20}\, 2^5(-x)^{20} + \ldots + (-x)^{25}$

The term containing x^{20} is ${}^{25}C_{20} \times 2^5 \times (-x)^{20}$.

$\qquad = 53\,130 \times 32 \times x^{20}$

$\qquad = 1\,700\,160x^{20}$

So the coefficient of x^{20} is 1 700 160.

Using the binomial theorem,

$$(1+x)^7 = 1^7 + {}^7C_1\, 1^6\, x + {}^7C_2\, 1^5\, x^2 + {}^7C_3\, 1^4\, x^3 + {}^7C_4\, 1^3 x^4 + \ldots$$
$$= 1 + {}^7C_1\, x + {}^7C_2\, x^2 + {}^7C_3\, x^3 + {}^7C_4\, x^4 + \ldots$$

But ${}^7C_1$, ${}^7C_2$, ${}^7C_3$ and ${}^7C_4$ can also be written as:

$${}^7C_1 = \frac{7!}{1!6!} = 7 \qquad {}^7C_2 = \frac{7!}{2!5!} = \frac{7\times 6}{2!} \qquad {}^7C_3 = \frac{7!}{3!4!} = \frac{7\times 6\times 5}{3!}$$

$${}^7C_4 = \frac{7!}{4!3!} = \frac{7\times 6\times 5\times 4}{4!}$$

So, $(1+x)^7 = 1 + 7x + \dfrac{7\times 6}{2!}x^2 + \dfrac{7\times 6\times 5}{3!}x^3 + \dfrac{7\times 6\times 5\times 4}{4!}x^4 + \ldots$

This leads to an alternative formula for binomial expansions:

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \frac{n(n-1)(n-2)(n-3)}{4!}x^4 + \ldots$$

The following example illustrates how this alternative formula can be applied.

WORKED EXAMPLE 5

Find the first four terms of the binomial expansion to

a $(1+3y)^7$ **b** $(2-y)^6$

Answers

a $(1+3y)^7 = 1 + 7(3y) + \dfrac{7\times 6}{2!}(3y)^2 + \dfrac{7\times 6\times 5}{3!}(3y)^3 + \ldots$

Replace x by $3y$ and n by 7 in the formula.

$= 1 + 21y + 189y^2 + 945y^3 + \ldots$

b $(2-y)^6 = \left[2\left(1-\dfrac{y}{2}\right)\right]^6$

The formula is for $(1+x)^n$ so take out a factor of 2.

$= 2^6\left(1-\dfrac{y}{2}\right)^6$

$= 2^6\left[1 + 6\left(-\dfrac{y}{2}\right) + \dfrac{6\times 5}{2!}\left(-\dfrac{y}{2}\right)^2 + \dfrac{6\times 5\times 4}{3!}\left(-\dfrac{y}{2}\right)^3 + \ldots\right]$

Replace x by $\left(-\dfrac{y}{2}\right)$ and n by 6 in the formula.

$= 2^6\left[1 - 3y + \dfrac{15}{4}y^2 - \dfrac{5}{2}y^3 + \ldots\right]$

Multiply terms in brackets by 2^6.

$= 64 - 192y + 240y^2 - 160y^3 + \ldots$

Exercise 11.2

1 Write the following rows of Pascal's triangle using combination notation.

a row 3 **b** row 4 **c** row 5

2 Use the binomial theorem to find the expansions of:

a $(1+x)^4$ **b** $(1-x)^5$ **c** $(1+2x)^4$ **d** $(3+x)^3$

e $(x+y)^4$ **f** $(2-x)^5$ **g** $(a-2b)^4$ **h** $(2x+3y)^4$

i $\left(\frac{1}{2}x-3\right)^4$ **j** $\left(1-\frac{x}{10}\right)^5$ **k** $\left(x-\frac{3}{x}\right)^5$ **l** $\left(x^2+\frac{1}{2x^2}\right)^6$.

3 Find the term in x^3 for each of the following expansions:

a $(2+x)^5$ **b** $(5+x)^8$ **c** $(1+2x)^6$ **d** $(3+2x)^5$

e $(1-x)^6$ **f** $(2-x)^9$ **g** $(10-3x)^7$ **h** $(4-5x)^{15}$.

4 Use the binomial theorem to find the first three terms in each of these expansions:

a $(1+x)^{10}$ **b** $(1+2x)^8$ **c** $(1-3x)^7$ **d** $(3+2x)^6$

e $(3-x)^9$ **f** $\left(2+\frac{1}{2}x\right)^8$ **g** $(5-x^2)^9$ **h** $(4x-5y)^{10}$.

5 **a** Write down, in ascending powers of x, the first 4 terms in the expansion of $(1+2x)^6$.

b Find the coefficient of x^3 in the expansions of $\left(1-\frac{x}{3}\right)(1+2x)^6$.

6 **a** Write down, in ascending powers of x, the first 4 terms in the expansion of $\left(1+\frac{x}{2}\right)^{13}$.

b Find the coefficient of x^3 in the expansions of $(1+3x)\left(1+\frac{x}{2}\right)^{13}$.

7 **a** Write down, in ascending powers of x, the first 4 terms in the expansion of $(1-3x)^{10}$.

b Find the coefficient of x^3 in the expansions of $(1-4x)(1-3x)^{10}$.

8 **a** Find, in ascending powers of x, the first 3 terms of the expansion of $(1+2x)^7$.

b Hence find the coefficient of x^2 in the expansion of $(1+2x)^7(1-3x+5x^2)$.

9 **a** Find, in ascending powers of x, the first 4 terms of the expansion of $(1+x)^7$.

b Hence find the coefficient of y^3 in the expansion of $(1+y-y^2)^7$.

10 Find the coefficient of x in the binomial expansion of $\left(x-\frac{3}{x}\right)^7$.

11 Find the term independent of x in the binomial expansion of $\left(x+\frac{1}{2x^2}\right)^9$.

CHALLENGE Q

12 When $(1+ax)^n$ is expanded the coefficients of x^2 and x^3 are equal.

Find a in terms of n.

11.3 Arithmetic progressions

At IGCSE level you learnt that a number sequence is an ordered set of numbers that satisfy a rule and that the numbers in the sequence are called the terms of the sequence. A number sequence is also called a **progression**.

The sequence 5, 8, 11, 14, 17, … is called an **arithmetic progression**. Each term differs from the term before by a constant. This constant is called the **common difference**.

The notation used for arithmetic progressions is:

a = first term $\quad d$ = common difference $\quad l$ = last term

The first five terms of an arithmetic progression whose first term is a and whose common difference is d are:

a	$a + d$	$a + 2d$	$a + 3d$	$a + 4d$
term 1	term 2	term 3	term 4	term 5

This leads to the formula:

$$n\text{th term} = a + (n - 1)d$$

WORKED EXAMPLE 6

Find the number of terms in the arithmetic progression –17, –14, –11, –8, …, 58.

Answers

$n\text{th term} = a + (n - 1)d$ — use $a = -17$, $d = 3$ and nth term = 58

$58 = -17 + 3(n - 1)$ — solve

$n - 1 = 25$

$n = 26$

WORKED EXAMPLE 7

The fifth term of an arithmetic progression is 4.4 and the ninth term is 7.6. Find the first term and the common difference.

Answers

5th term = 4.4 $\quad \Rightarrow a + 4d = 4.4$ -------(1)

9th term = 7.6 $\quad \Rightarrow a + 8d = 7.6$ -------(2)

(2) – (1), gives $4d = 3.2$

$d = 0.8$

Substituting in (1) gives $a + 3.2 = 4.4$

$a = 1.2$

First term = 1.2, common difference = 0.8.

WORKED EXAMPLE 8

The nth term of an arithmetic progression is $11 - 3n$. Find the first term and the common difference.

Answers

1st term $= 11 - 3(1) = 8$	substitute $n = 1$ into nth term $= 11 - 3n$
2nd term $= 11 - 3(2) = 5$	substitute $n = 2$ into nth term $= 11 - 3n$

Common difference = 2nd term − 1st term = −3.

The sum of an arithmetic progression

When the terms in a sequence are added together the resulting sum is called a **series**.

CLASS DISCUSSION

$1 + 2 + 3 + 4 + \ldots + 97 + 98 + 99 + 100 = ?$

It is said that at the age of eight, the famous mathematician Carl Gauss was asked to find the sum of the numbers from 1 to 100. His teacher expected this task to keep him occupied for some time but Gauss surprised his teacher by writing down the correct answer after just a couple of seconds. His method involved adding the numbers in pairs: $1 + 100 = 101$, $2 + 99 = 101$, $3 + 98 = 101$, …

1 Can you complete his method to find the answer?

2 Use Gauss's method to find the sum of

a $2 + 4 + 6 + 8 + \ldots + 394 + 396 + 398 + 400$

b $3 + 6 + 9 + 12 + \ldots + 441 + 444 + 447 + 450$

c $17 + 24 + 31 + 15 + \ldots + 339 + 346 + 353 + 360$.

3 Use Gauss's method to find an expression, in terms of n, for the sum

$1 + 2 + 3 + 4 + \ldots + (n - 3) + (n - 2) + (n - 1) + n$.

It can be shown that the sum of an arithmetic progression, S_n, can be written as:

$$S_n = \frac{n}{2}(a + l) \quad \text{or} \quad S_n = \frac{n}{2}[2a + (n - 1)d]$$

Proof: $S_n = a + (a + d) + (a + 2d) + \ldots + (l - 2d) + (l - d) + l$

Reversing: $S_n = l + (l - d) + (l - 2d) + \ldots + (a + 2d) + (a + d) + a$

Adding: $2S_n = n(a + l) + (a + l) + (a + l) + \ldots + (a + l) + (a + l) + (a + l) +$

$2S_n = n(a + l)$

$$S_n = \frac{n}{2}(a + l)$$

Using $l = a + (n-1)d$, gives $S_n = \frac{n}{2}[2a + (n-1)d]$

It is useful to remember the following rule that applies for all progressions:

$$n\text{th term} = S_n - S_{n-1}$$

WORKED EXAMPLE 9

In an arithmetic progression, the 1st term is 25, the 19th term is –38 and the last term is –87. Find the sum of all the terms in the progression.

Answers

nth term $= a + (n-1)d$ — use nth term $= -38$ when $n = 19$ and $a = 25$

$-38 = 25 + 18d$ — solve

$d = -3.5$

nth term $= a + (n-1)d$ — use nth term $= -87$ when $a = 25$ and $d = -3.5$

$-87 = 25 - 3.5(n-1)$ — solve

$n - 1 = 32$

$n = 33$

$S_n = \frac{n}{2}(a + l)$ — use $a = 25$, $l = -87$ and $n = 33$

$S_{33} = \frac{33}{2}(25 - 87)$

$= -1023$

WORKED EXAMPLE 10

The 12th term in an arithmetic progression is 8 and the sum of the first 13 terms is 78. Find the first term of the progression and the common difference.

Answers

nth term $= a + (n-1)d$ — use nth term $= 8$ when $n = 12$

$8 = a + 11d$ ------------ (1)

$S_n = \frac{n}{2}[2a + (n-1)d]$ — use $n = 13$ and $S_{13} = 78$

$78 = \frac{13}{2}(2a + 12d)$ — simplify

$6 = a + 6d$ --------------------- (2)

(1) – (2) gives $5d = 2$

$d = 0.4$

Substituting $d = 0.4$ in equation (1) gives $a = 3.6$.

First term = 3.6, common difference = 0.4.

WORKED EXAMPLE 11

The sum of the first n terms, S_n, of a particular arithmetic progression is given by $S_n = 5n^2 - 3n$.

a Find the first term and the common difference.

b Find an expression for the nth term.

Answers

a $S_1 = 5(1)^2 - 3(1) = 2 \quad \Rightarrow \quad$ first term = 2

$S_2 = 5(2)^2 - 3(2) = 14 \quad \Rightarrow \quad$ first term + second term = 14

second term = 14 – 2 = 12

First term = 2, common difference = 10.

b **Method 1:**

$$\begin{aligned} n\text{th term} &= a + (n-1)d \qquad \text{use } a = 2,\ d = 10 \\ &= 2 + 10(n-1) \\ &= 10n - 8 \end{aligned}$$

Method 2:

$$\begin{aligned} n\text{th term} &= S_n - S_{n-1} = 5n^2 - 3n - [5(n-1)^2 - 3(n-1)] \\ &= 5n^2 - 3n - (5n^2 - 10n + 5 - 3n + 3) \\ &= 10n - 8 \end{aligned}$$

Exercise 11.3

1 The first term in an arithmetic progression is a and the common difference is d.

Write down expressions, in terms of a and d, for the 5th term and the 14th term.

2 Find the sum of each of these arithmetic series.

a $2 + 9 + 16 + \ldots$ (15 terms) **b** $20 + 11 + 2 + \ldots$ (20 terms)

c $8.5 + 10 + 11.5 + \ldots$ (30 terms) **d** $-2x - 5x - 8x - \ldots$ (40 terms)

3 Find the number of terms and the sum of each of these arithmetic series.

a $23 + 27 + 31 \ldots + 159$ **b** $28 + 11 - 6 - \ldots -210$

4 The first term of an arithmetic progression is 2 and the sum of the first 12 terms is 618.

Find the common difference.

5 In an arithmetic progression, the 1st term is -13, the 20th term is 82 and the last term is 112.

a Find the common difference and the number of terms.

b Find the sum of the terms in this progression.

6 The first two terms in an arithmetic progression are 57 and 46. The last term is -207. Find the sum of all the terms in this progression.

7 The first two terms in an arithmetic progression are –2 and 5. The last term in the progression is the only number in the progression that is greater than 200. Find the sum of all the terms in the progression.

8 The first term of an arithmetic progression is 8 and the last term is 34. The sum of the first six terms is 58. Find the number of terms in this progression.

9 Find the sum of all the integers between 100 and 400 that are multiples of 6.

10 The first term of an arithmetic progression is 7 and the eleventh term is 32. The sum of all the terms in the progression is 2790. Find the number of terms in the progression.

11 Rafiu buys a boat for \$15 500. He pays for this boat by making monthly payments that are in arithmetic progression. The first payment that he makes is \$140 and the debt is fully repaid after 31 payments. Find the fifth payment.

12 The eighth term of an arithmetic progression is −10 and the sum of the first twenty terms is −350.

a Find the first term and the common difference.

b Given that the nth term of this progression is −97, find the value of n.

13 The sum of the first n terms, S_n, of a particular arithmetic progression is given by $S_n = 4n^2 + 2n$. Find the first term and the common difference.

14 The sum of the first n terms, S_n, of a particular arithmetic progression is given by $S_n = -3n^2 - 2n$. Find the first term and the common difference.

15 The sum of the first n terms, S_n, of a particular arithmetic progression is given by $S_n = \frac{n}{12}(4n+5)$. Find an expression for the nth term.

16 A circle is divided into twelve sectors. The sizes of the angles of the sectors are in arithmetic progression. The angle of the largest sector is 6.5 times the angle of the smallest sector. Find the angle of the smallest sector.

17 An arithmetic sequence has first term a and common difference d. The sum of the first 25 terms is 15 times the sum of the first 4 terms.

a Find d in terms of a.

b Find the 55th term in terms of a.

18 The eighth term in an arithmetic progression is three times the third term. Show that the sum of the first eight terms is four times the sum of the first four terms.

CHALLENGE Q

19 The first term of an arithmetic progression is $\cos^2 x$ and the second term is 1.

a Write down an expression, in terms of $\cos x$, for the seventh term of this progression.

b Show that the sum of the first twenty terms of this progression is $20 + 170\sin^2 x$.

CHALLENGE Q

20 The sum of the digits in the number 56 is 11. $(5 + 6 = 11)$

a Show that the sum of the digits of the integers from 15 to 18 is 30.

b Find the sum of the digits of the integers from 1 to 100.

11.4 Geometric progressions

The sequence 7, 14, 28, 56, 112, … is called a **geometric progression**. Each term is double the preceding term. The constant multiple is called the **common ratio**.

The notation used for a geometric progression is:

a = first term $\quad\quad r$ = common ratio

The first five terms of a geometric progression whose first term is a and whose common ratio is r are:

a	ar	ar^2	ar^3	ar^4
term 1	term 2	term 3	term 4	term 5

This leads to the formula:

$$n\text{th term} = ar^{n-1}$$

WORKED EXAMPLE 12

The 3rd term of a geometric progression is 144 and the common ratio is $\frac{3}{2}$. Find the 7th term and an expression for the nth term.

Answers

$n\text{th term} = ar^{n-1}$ $\quad\quad$ use nth term = 144 when $n = 3$ and $r = \frac{3}{2}$

$144 = a\left(\frac{3}{2}\right)^2$

$a = 64$

$7\text{th term} = 64\left(\frac{3}{2}\right)^6 = 729$

$n\text{th term} = ar^{n-1} = 64\left(\frac{3}{2}\right)^{n-1}$

WORKED EXAMPLE 13

The 2nd and 4th terms in a geometric progression are 108 and 48 respectively. Given that all the terms are positive, find the 1st term and the common ratio. Hence, write down an expression for the nth term.

Answers

$108 = ar$---------(1) $48 = ar^3$----------(2)

(2) ÷ (1) gives $\dfrac{ar^3}{ar} = \dfrac{48}{108}$

$r^2 = \dfrac{4}{9}$

$r = \pm\dfrac{2}{3}$ all terms are positive n ⇒ > 0

$r = \dfrac{2}{3}$

Substituting $r = \dfrac{2}{3}$ into equation (1) gives $a = 162$.

First term = 162, common ratio $= \dfrac{2}{3}$, nth term $= 162\left(\dfrac{2}{3}\right)^{n-1}$.

WORKED EXAMPLE 14

The nth term of a geometric progression is $30\left(-\dfrac{1}{2}\right)^n$. Find the first term and the common ratio.

Answers

1st term $= 30\left(-\dfrac{1}{2}\right)^1 = -15$

2nd term $= 30\left(-\dfrac{1}{2}\right)^2 = 7.5$

Common ratio $= \dfrac{\text{2nd term}}{\text{1st term}} = \dfrac{7.5}{-15} = -\dfrac{1}{2}$

First term $= -15$, common ratio $= -\dfrac{1}{2}$.

WORKED EXAMPLE 15

In the geometric sequence 2, 6, 18, 54, … which is the first term to exceed 1000000?

Answers

$n\text{th term} = ar^{n-1}$	use $a = 2$ and $r = 3$
$23^{n-1} > 1000000$	divide by 2 and take logs
$\log_{10} 3^{n-1} > \log10\ 500000$	use the power rule for logs
$(n-1)\log_{10} 3 > \log_{10} 500000$	divide both sides by $\log_{10} 3$
$n - 1 > \dfrac{\log_{10} 500000}{\log_{10} 3}$	
$n - 1 > 11.94\ldots$	
$n > 12.94\ldots$	

The 13th term is the first to exceed 1000000.

CLASS DISCUSSION

In this class discussion you are not allowed to use a calculator.

1 Consider the sum of the first 10 terms, S_{10}, of a geometric progression with $a = 1$ and $r = 5$.

$S_{10} = 1 + 5 + 5^2 + 5^3 + \ldots + 5^7 + 5^8 + 5^9$

- **a** Multiply both sides of the equation above by the common ratio, 5, and complete the following statement.

  $5S_{10} = 5 + 5^2 + 5^{\cdots} + 5^{\cdots} + \ldots + 5^{\cdots} + 5^{\cdots} + 5^{\cdots}$
- **b** What happens when you subtract the equation for S_{10} from the equation for $5S_{10}$?
- **c** Can you find an alternative way of expressing the sum S_{10}?

2 Use the method from **question 1** to find an alternative way of expressing each of the following:

- **a** $3 + 32 + 32^2 + 32^3 + \ldots$ (12 terms)
- **b** $32 + 32\dfrac{1}{2} + 32\left(\dfrac{1}{2}\right)^2 + 32\left(\dfrac{1}{2}\right)^3 + \ldots$ (15 terms)
- **c** $27 - 18 + 12 - 8 + \ldots$ (20 terms)

It can be shown that the sum of a geometric progression, S_n, can be written as:

$$S_n = \frac{a(1 - r^n)}{1 - r} \quad \text{or} \quad S_n = \frac{a(r^n - 1)}{r - 1}$$

Note:
For these formulae, $r \neq 1$

Either formula can be used but it is usually easier to

- use the first formula when $-1 < r < 1$.
- use the second formula when $r > 1$ or when $r < -1$.

Proof:

$$Sn = a + ar + ar^2 + \ldots + ar^{n-3} + ar^{n-2} + ar^{n-1} \quad \text{-------------(1)}$$

$r \times (1)$: $$rS_n = ar + ar^2 + \ldots + ar^{n-3} + ar^{n-2} + ar^{n-1} + ar^n \quad \text{-------------(2)}$$

$(2) - (1)$: $$rS_n - S_n = ar^n - a$$

$$(r-1)S_n = a(r^n - 1)$$

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

Multiplying numerator and denominator by -1 gives the alternative formula $S_n = \dfrac{a(1 - r^n)}{1 - r}$.

WORKED EXAMPLE 16

Find the sum of the first ten terms of the geometric series $2 + 6 + 18 + 54 + \ldots$

Answers

$S_n = \dfrac{a(r^n - 1)}{r - 1}$ use $a = 2$, $r = 3$ and $n = 10$

$S_{12} = \dfrac{2(3^{10} - 1)}{3 - 1}$ simplify

$= 59\,048$

WORKED EXAMPLE 17

The second term of a geometric progression is 9 less than the first term. The sum of the second and third terms is 30. Given that all the terms in the progression are positive, find the first term.

Answers

2nd term = 1st term -9

$ar = a - 9$ rearrange to make a the subject

$a = \dfrac{9}{1 - r}$ ——(1)

2nd term + 3rd term $= 30$

$ar + ar^2 = 30$ factorise

$ar(1 + r) = 30$ ——(2)

(2) ÷ (1) gives $\dfrac{ar(1 + r)}{a} = \dfrac{30(1 - r)}{9}$ simplify

$3r^2 + 13r - 10 = 0$ factorise and solve

$(3r - 2)(r + 5) = 0$

$r = \dfrac{2}{3}$ or $r = -5$ all terms are positive $\Rightarrow r > 0$

$r = \dfrac{2}{3}$

Substituting $r = \dfrac{2}{3}$ into (1) gives $a = 27$.

First term is 27.

Exercise 11.4

1 Identify whether the following sequences are geometric. If they are geometric, write down the common ratio and the eighth term.

a $1, 2, 4, 6, \ldots$

b $-1, 4, -16, 64, \ldots$

c $81, 27, 9, 3, \ldots$

d $\frac{2}{11}, \frac{3}{11}, \frac{5}{11}, \frac{8}{11}, \ldots$

e $2, 0.4, 0.08, 0.16, \ldots$

f $-5, 5, -5, 5, \ldots$

2 The first term in a geometric progression is a and the common ratio is r. Write down expressions, in terms of a and r, for the 9th term and the 20th term.

3 The 3rd term of a geometric progression is 108 and the 6th term is -32. Find the common ratio and the first term.

4 The first term of a geometric progression is 75 and the third term is 27. Find the two possible values for the fourth term.

5 The second term of a geometric progression is 12 and the fourth term is 27. Given that all the terms are positive, find the common ratio and the first term.

6 The 6th and 13th terms of a geometric progression are $\frac{5}{2}$ and 320 respectively. Find the common ratio, the first term and the 10th term of this progression.

7 The sum of the second and third terms in a geometric progression is 30. The second term is 9 less than the first term. Given that all the terms in the progression are positive, find the first term.

8 Three consecutive terms of a geometric progression are x, $x + 6$ and $x + 9$. Find the value of x.

9 In the geometric sequence $\frac{1}{4}, \frac{1}{2}, 1, 2, 4, \ldots$ which is the first term to exceed 500000?

10 In the geometric sequence $256, 128, 64, 32, \ldots$ which is the first term that is less than 0.001?

11 Find the sum of the first eight terms of each of these geometric series.

a $4 + 8 + 16 + 32 + \ldots$

b $729 + 243 + 81 + 27 + \ldots$

c $2 - 6 + 18 - 54 + \ldots$

d $-500 + 1000 - 200 + 40 - \ldots \ldots$

12 The first four terms of a geometric progression are 1, 3, 9 and 27. Find the smallest number of terms that will give a sum greater than 2 000 000.

13 A ball is thrown vertically upwards from the ground. The ball rises to a height of 10 m and then falls and bounces. After each bounce it rises to $\frac{4}{5}$ of the height of the previous bounce.

a Write down an expression, in terms of n, for the height that the ball rises after the nth impact with the ground.

b Find the total distance that the ball travels from the first throw to the fifth impact with the ground.

14 The third term of a geometric progression is nine times the first term. The sum of the first four terms is k times the first term. Find the possible values of k.

15 John competes in a 10 km race. He completes the first kilometre in 4 minutes. He reduces his speed in such a way that each kilometre takes him 1.05 times the time taken for the preceding kilometre. Find the total time, in minutes and seconds, John takes to complete the 10 km race.

Give your answer correct to the nearest second.

16 A geometric progression has first term a, common ratio r and sum to n terms, S_n.

Show that $\dfrac{S_{3n} - S_{2n}}{S_n} = r^{2n}$.

CHALLENGE Q

17 $1, 1, 3, \frac{1}{3}, 9, \frac{1}{9}, 27, \frac{1}{27}, 81, \frac{1}{81}, \ldots$

Show that the sum of the first $2n$ terms of this sequence is $\frac{1}{2}\left(3^n - 3^{1-n} + 2\right)$.

CHALLENGE Q

18 $S_n = 6 + 66 + 666 + 6666 + 66666 + \ldots$

Find the sum of the first n terms of this sequence.

11.5 Infinite geometric series

An infinite series is a series whose terms continue forever.
The geometric series where $a = 2$ and $r = \frac{1}{2}$ is $2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$

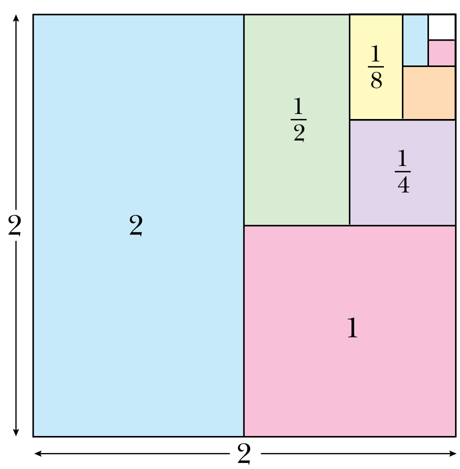
For this series it can be shown that

$S_1 = 2,\ S_2 = 3,\ S_3 = 3\frac{1}{2},\ S_4 = 3\frac{3}{4},\ S_5 = 3\frac{7}{8}, \ldots .$

This suggests that the sum to infinity approaches the number 4.

The diagram of the 2 by 2 square is a visual representation of this series. If the pattern of rectangles inside the square is continued the total areas of the inside rectangles approaches the value 4.

This confirms that the sum to infinity of the series $2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$ is 4.



This is an example of a **convergent** series because the sum to infinity converges on a finite number.

CLASS DISCUSSION

1 Use a spread sheet to investigate whether the sum of each of these infinite geometric series converge or diverge. If they converge, state their sum to infinity.

$a = \frac{2}{5},\ r = 2$ $\quad$ $a = -3,\ r = -\frac{1}{2}$ $\quad$ $a = 5,\ r = \frac{2}{3}$ $\quad$ $a = \frac{1}{2},\ 5 = -5$

2 Find other convergent geometric series of your own. In each case find the sum to infinity.

3 Can you find a condition for r for which a geometric series is convergent?

Consider the geometric series $a + ar + ar^2 + ar^3 + \ \ldots \ + ar^{n-1}$.

The sum, S_n, is given by the formula $S_n = \dfrac{a(1 - r^n)}{1 - r}$.

If $-1 < r < 1$, then as n gets larger and larger, r^n gets closer and closer to 0.

We say that as $n \to \infty,\ r^n \to 0$.

Hence, as $n \to \infty$, $\dfrac{a(1 - r^n)}{1 - r} \to \dfrac{a(1 - 0)}{1 - r} = \dfrac{a}{1 - r}$.

This gives the result

$$S_\infty = \frac{a}{1 - r} \quad \text{provided that } -1 < r\ 1 - 1$$

Note:
This is not true when $r > 1$ or when $r = -1$

WORKED EXAMPLE 18

The first three terms of a geometric progression are 25, 15 and 9.

a Write down the common ratio.

b Find the sum to infinity.

a Common ratio $\dfrac{\text{second term}}{\text{firest term}} = \dfrac{15}{25} = \dfrac{3}{5}$

b $S_\infty = \dfrac{a}{1 - r}$ $\qquad$ use $a = 25$ and $r = \dfrac{3}{5}$

$= \dfrac{25}{1 - \frac{3}{5}}$

$= 62.5$

WORKED EXAMPLE 19

A geometric progression has a common ratio of $-\frac{4}{5}$ and the sum of the first four terms is 164.

a Find the first term of the progression.

b Find the sum to infinity.

a $S_4 = \frac{a(1-r^4)}{1-r}$ — use $S_4 = 164$ and $r = -\frac{4}{5}$

$164 = \frac{a\left(1-\left(-\frac{4}{5}\right)^4\right)}{1-\left(-\frac{4}{5}\right)}$ — simplify

$164 = \frac{41}{125}a$ — solve

$a = 500$

b $S_\infty = \frac{a}{1-r}$ — use $a = 500$ and $r = -\frac{4}{5}$

$= \frac{500}{1-\left(-\frac{4}{5}\right)}$

$= 277\frac{7}{9}$

Exercise 11.5

1 Find the sum to infinity of each of the following geometric series.

a $3+1+\frac{1}{3}+\frac{1}{9}+\ldots$

b $1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{16}-\ldots$

c $8+\frac{8}{5}+\frac{8}{25}+\frac{8}{125}+\ldots$

d $-162+108-72+48-\ldots$

2 The first term of a geometric progression is 10 and the second term is 8. Find the sum to infinity.

3 The first term of a geometric progression is 300 and the fourth term is $-2\frac{2}{5}$. Find the common ratio and the sum to infinity.

4 The first four terms of a geometric progression are 1, 0.8^2, 0.8^4 and 0.8^6. Find the sum to infinity.

5 **a** Write the recurring decimal $0.\dot{4}\dot{2}$ as the sum of a geometric progression.

b Use your answer to **part a** to show that $0.\dot{4}\dot{2}$ can be written as $\frac{14}{33}$.

6 The first term of a geometric progression is –120 and the sum to infinity is –72. Find the common ratio and the sum of the first three terms.

7 The second term of a geometric progression is 6.5 and the sum to infinity is 26. Find the common ratio and the first term.

8 The second term of a geometric progression is –96 and the fifth term is $40\frac{1}{2}$.

a Find the common ratio and the first term.

b Find the sum to infinity.

9 The first three terms of a geometric progression are 175, k and 63. Given that all the terms in the progression are positive, find:

a the value of k

b the sum to infinity.

10 The second term of a geometric progression is 18 and the fourth term is 1.62. Given that the common ratio is positive, find:

a the common ratio and the first term

b the sum to infinity.

11 The first three terms of a geometric progression are $k + 15$, k and $k - 12$ respectively, find:

a the value of k

b the sum to infinity.

12 The fourth term of a geometric progression is 48 and the sum to infinity is three times the first term. Find the first term.

13 A geometric progression has first term a and common ratio r. The sum of the first three terms is 62 and the sum to infinity is 62.5. Find the value of a and the value of r.

14 The first term of a geometric progression is 1 and the second term is $2\sin x$ where $-\frac{\pi}{2} < x < \frac{\pi}{2}$. Find the set of values of x for which this progression is convergent.

15 A ball is dropped from a height of 12 m. After each bounce it rises to $\frac{3}{4}$ of the height of the previous bounce. Find the total vertical distance that the ball travels.

CHALLENGE Q

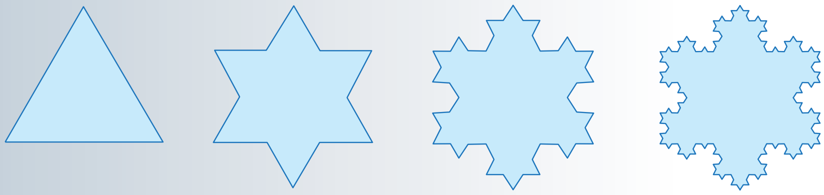
16 Starting with an equilateral triangle, a Koch snowflake pattern can be constructed using the following steps:

Step 1: Divide each line segment into three equal segments.

Step 2: Draw an equilateral triangle, pointing outwards, which has the middle segment from step 1 as its base.

Step 3: Remove the line segments that were used as the base of the equilateral triangles in step 2.

These three steps are then repeated to produce the next pattern.



Pattern 1 *Pattern 2* *Pattern 3* *Pattern 4*

You are given that the triangle in **pattern 1** has side length x units.

a Find, in terms of x, expressions for the perimeter of each of patterns 1, 2, 3 and 4 and explain why this progression for the perimeter of the snowflake diverges to infinity.

b Show that the area of each of patterns 1, 2, 3 and 4 can be written as:

Pattern	Area
1	$\frac{\sqrt{3}x^2}{4}$
2	$\frac{\sqrt{3}x^2}{4} + 3\frac{\sqrt{3}\left(\frac{x}{3}\right)^2}{4}$
3	$\frac{\sqrt{3}x^2}{4} + 3\frac{\sqrt{3}\left(\frac{x}{3}\right)^2}{4} + 12\frac{\sqrt{3}\left(\frac{x}{9}\right)^2}{4}$
4	$\frac{\sqrt{3}x^2}{4} + 3\frac{\sqrt{3}\left(\frac{x}{3}\right)^2}{4} + 12\frac{\sqrt{3}\left(\frac{x}{9}\right)^2}{4} + 48\frac{\sqrt{3}\left(\frac{x}{27}\right)^2}{4}$

Hence show that the progression for the area of the snowflake converges to $\frac{8}{5}$ times the area of the original triangle.

CHALLENGE Q

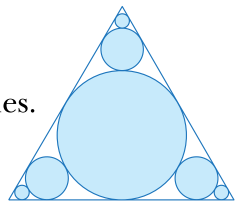
17 A circle of radius 1 unit is drawn touching the three edges of an equilateral triangle.

Three smaller circles are then drawn at each corner to touch the original circle and two edges of the triangle.

This process is then repeated an infinite number of times.

a Find the sum of the circumferences of all the circles.

b Find the sum of the areas of all the circles.



11.6 Further arithmetic and geometric series

Some problems may involve more than one progression.

CLASS DISCUSSION

a, b, c, …

1 Given that a, b and c are in arithmetic progression, find an equation connecting a, b and c.

2 Given that a, b and c are in geometric progression, find an equation connecting a, b and c.

WORKED EXAMPLE 20

The first, second and third terms of an arithmetic series are x, y and x^2. The first, second and third terms of a geometric series are x, x^2 and y. Given that $x < 0$, find:

a the value of x and the value of y

b the sum to infinity of the geometric series

c the sum of the first 20 terms of the arithmetic series.

a Arithmetic series is: $x + y + x^2 + \dots \dots$ — use common differences

$$y - x = x^2 - y$$
$$2y = x^2 + x \text{-----------}(1)$$

Geometric series is: $x + x^2 + y + \dots \dots$ — use common ratios

$$\frac{y}{x^2} = \frac{x^2}{x}$$
$$y = x^3 \text{-----------------}(2)$$

(1) and (2) give $2x^3 = x^2 + x$ — divide by x (since $x \neq 0$) and rearrange

$2x^2 - x - 1 = 0$ — factorise and solve

$(2x + 1)(x - 1) = 0$

$x = -\dfrac{1}{2}$ or $x = 1$ — $x \neq 1$ since $x < 0$

Hence, $x = -\dfrac{1}{2}$ and $y = -\dfrac{1}{8}$.

b $S_\infty = \dfrac{a}{1-r}$ use $a = -\dfrac{1}{2}$ and $r = -\dfrac{1}{2}$

$S_\infty = \dfrac{-\frac{1}{2}}{1-\left(-\frac{1}{2}\right)} = -\dfrac{1}{3}$

c $S_n = \dfrac{n}{2}[2a + (n-1)d]$ use $n = 20$, $a = -\dfrac{1}{2}$, $d = y - x = \dfrac{3}{8}$

$S_{20} = \dfrac{20}{2}\left[-1 + 19\left(\dfrac{3}{8}\right)\right]$

$= 61.25$

Exercise 11.6

1 The first term of a progression is 8 and the second term is 12. Find the sum of the first six terms given that the progression is:

a arithmetic **b** geometric.

2 The first term of a progression is 25 and the second term is 20.

a Given that the progression is geometric, find the sum to infinity.

b Given that the progression is arithmetic, find the number of terms in the progression if the sum of all the terms is –1550.

3 The 1st, 2nd and 3rd terms of a geometric progression are the 1st, 5th and 11th terms respectively of an arithmetic progression. Given that the first term in each progression is 48 and the common ratio of the geometric progression is r, where $r \neq 1$, find:

a the value of r,

b the 6th term of each progression.

4 A geometric progression has six terms. The first term is 486 and the common ratio is $\dfrac{2}{3}$. An arithmetic progression has 35 terms and common difference $\dfrac{3}{2}$. The sum of all the terms in the geometric progression is equal to the sum of all the terms in the arithmetic progression. Find the first term and the last term of the arithmetic progression.

5 The 1st, 2nd and 3rd terms of a geometric progression are the 1st, 5th and 8th terms respectively of an arithmetic progression. Given that the first term in each progression is 200 and the common ratio of the geometric progression is r, where $r \neq 1$ find:

a the value of r,

b the 4th term of each progression,

c the sum to infinity of the geometric progression.

6 The first term of an arithmetic progression is 12 and the sum of the first 16 terms is 282.

a Find the common difference of this progression.

The 1st, 5th and nth term of this arithmetic progression are the 1st, 2nd and 3rd term respectively of a geometric progression.

b Find the common ratio of the geometric progression and the value of n.

7 The first two terms of a geometric progression are 80 and 64 respectively. The first three terms of this geometric progression are also the 1st, 11th and nth terms respectively of an arithmetic progression. Find the value of n.

8 The first two terms of a progression are $5x$ and x^2 respectively.

a For the case where the progression is arithmetic with a common difference of 24, find the two possible values of x and the corresponding values of the third term.

b For the case where the progression is geometric with a third term of $-\frac{8}{5}$, find the common ratio and the sum to infinity.

Summary

Binomial expansions

If n is a positive integer then $(a+b)^n$ can be expanded using the formula

$$(a+b)^n = a^n + {}^nC_1\, a^{n-1}\, b + {}^nC_2\, a^{n-2}\, b^2 + {}^nC_3\, a^{n-3}\, b^3 + \ldots + {}^nC_r\, a^{n-r}\, b^r + \ldots + b^n$$

or

$$(a+b)^n = a^n + \binom{n}{1} a^{n-1}\, b + \binom{n}{2} a^{n-2}\, b^2 + \binom{n}{3} a^{n-3}\, b^3 + \ldots + \binom{n}{r} a^{n-r}\, b^r + \ldots + b^n$$

and where ${}^nC_r = \binom{n}{r} = \frac{n!}{(n-r)!\,r!}$.

In particular,

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \frac{n(n-1)(n-2)(n-3)}{4!}x^4 + \ldots + x^n.$$

Arithmetic series

For an arithmetic progression with first term a, common difference d and n terms:

- the kth term $= a + (k-1)d$
- the last term $= l = a + (n-1)d$
- the sum of the terms $= S_n = \frac{n}{2}(a+l) = \frac{n}{2}[2a+(n-1)d]$.

Geometric series

For a geometric progression with first term a, common ratio r and n terms:

- the kth term $= ar^{k-1}$
- the last term $= ar^{n-1}$
- the sum of the terms $= S_n = \dfrac{a(1-r^n)}{1-r} = \dfrac{a(r^n-1)}{r-1}$.

The condition for a geometric series to converge is $-1 < r < 1$.

When a geometric series converges, $S_\infty = \dfrac{a}{1-r}$.

Examination questions

Worked past paper example

a Find the first 4 terms in the expansion of $(2+x^2)^6$ in ascending powers of x. [3]

b Find the term independent of x in the expansion of $(2+x^2)^6\left(1-\dfrac{3}{x^2}\right)^2$. [3]

Cambridge IGCSE Additional Mathematics 0606 Paper 11 Q3i,ii Jun 2015

Answer

a Expanding $(2+x^2)^6$ using the binomial theorem gives

$$2^6 + {}^6C_1\, 2^5 x^2 + {}^6C_2\, 2^4 (x^2)^2 + {}^6C_3\, 2^3 (x^2)^3 = 64 + 192x^2 + 240x^4 + 160x^6 \ldots$$

b $$(2+x^2)^6\left(1-\frac{2}{x^2}\right)^2 = (64+192x^2+240x^4+160x^6\ldots)\left(1-\frac{6}{x^2}+\frac{9}{x^4}\right)$$

$$\text{Term independent of } x = (64\times 1) + \left(192x^2 \times -\frac{6}{x^2}\right) + \left(240x^4 \times \frac{9}{x^4}\right)$$
$$= 64 - 1152 + 2160$$
$$= 1072$$

Exercise 11.7

Exam Exercise

1 **a** Find the first four terms in the expansion of $(2+x)^6$ in ascending powers of x. [3]

b Hence find the coefficient of x^3 in the expansion of $(1+3x)(1-x)(2+x)^6$. [4]

Cambridge IGCSE Additional Mathematics 0606 Paper 21 Q7i,ii Jun 2013

2 **a** Find the first 3 terms, in descending powers of x, in the expansion of $\left(x+\dfrac{2}{x^2}\right)^6$. [3]

b Hence find the term independent of x in the expansion of $\left(2-\dfrac{4}{x^3}\right)\left(x+\dfrac{2}{x^2}\right)^6$. [2]

Cambridge IGCSE Additional Mathematics 0606 Paper 11 Q6i,ii Nov 2012

3 The coefficient of x^2 in the expansion of $\left(1+\frac{x}{5}\right)^n$, where n is a positive integer is $\frac{3}{5}$.

a Find the value of n. [4]

b Using this value of n, find the term independent of x in the expansion of

$$\left(1+\frac{x}{5}\right)^n\left(2-\frac{3}{x}\right)^2.$$ [4]

Cambridge IGCSE Additional Mathematics 0606 Paper 11 Q7i,ii Nov 2011

4 **a** Find the coefficient of x^3 in the expansion of $\left(1-\frac{x}{2}\right)^{12}$. [2]

b Find the coefficient of x^3 in the expansion of $(1+4x)\left(1-\frac{x}{2}\right)^{12}$. [3]

Cambridge IGCSE Additional Mathematics 0606 Paper 21 Q2i,ii Jun 2011

5 **a** Find, in ascending powers of x, the first 3 terms in the expansion of $(2-5x)^6$, giving your answer in the form $a+bx+cx^2$, where a, b and c are integers. [3]

b Find the coefficient of x in the expansion of $(2-5x)^6\left(1+\frac{x}{2}\right)^{10}$. [3]

Cambridge IGCSE Additional Mathematics 0606 Paper 11 Q6i,ii Nov 2010

6 **i** Write down, in ascending powers of x, the first 3 terms in the expansion of $(3+2x)^6$. Give each term in its simplest form. [3]

ii Hence find the coefficient of x^2 in the expansion of $(2-x)(3+2x)^6$. [2]

Cambridge IGCSE Additional Mathematics 0606 Paper 12 Q4 Mar 2015

7 **i** Find the first 4 terms in the expansion of $(2+x^2)^6$ in ascending powers of x. [3]

ii Find the term independent of x in the expansion of $\left(2+x^2\right)^6\left(1-\frac{3}{x^2}\right)^2$. [3]

Cambridge IGCSE Additional Mathematics 0606 Paper 11 Q3 Jun 2015

8 **a** **i** Use the Binomial Theorem to expand $(a+b)^4$, giving each term in its simplest form. [2]

ii Hence find the term independent of x in the expansion of $\left(2x+\frac{1}{5x}\right)^4$. [2]

b The coefficient of x^3 in the expansion of $\left(1+\frac{x}{2}\right)^n$ equals $\frac{5n}{12}$. Find the value of the positive integer n. [3]

Cambridge IGCSE Additional Mathematics 0606 Paper 21 Q8 Jun 2016

9 The first term of a geometric progression is 35 and the second term is −14.

a Find the fourth term. [3]

b Find the sum to infinity. [2]

Examination style question

10 The first three terms of a geometric progression are $2k + 6$, $k + 12$ and k respectively. All the terms in the progression are positive.

a Find value of k. [3]

b Find the sum to infinity. [2]

Examination style question

11 An arithmetic progression has first term a and common difference d. Give that the sum of the first 100 terms is 25 times the sum of the first 20 terms.

a Find d in terms of a. [3]

b Write down an expression, in terms of a, for the 50th. [2]

Examination style question

12 The 15th term of an arithmetic progression is 3 and the sum of the first 8 terms is 194.

a Find the first term of the progression and the common difference. [4]

b Given that the nth term of the progression is –22, find the value of n. [2]

Examination style question

13 The second term of a geometric progression is –576 and the fifth term is 243. Find

a the common ratio [3]

b the first term [1]

c the sum to infinity. [2]

Examination style question

14 a The sixth term of an arithmetic progression is 35 and the sum of the first ten terms is 335. Find the eighth term. [4]

b A geometric progression has first term 8 and common ratio r. A second geometric progression has first term 10 and common ratio $\frac{1}{4}r$. The two progressions have the same sum to infinity, S. Find the values of r and the value of S. [3]

Examination style question

15 a The 10th term of an arithmetic progression is 4 and the sum of the first 7 terms is –28. Find the first term and the common difference. [4]

b The first term of a geometric progression is 40 and the fourth term is 5. Find the sum to infinity of the progression. [3]

Examination style question

16 a A geometric progression has first term a, common ratio r and sum to infinity S.

A second geometric progression has first term $3a$, common ratio $2r$ and sum to infinity $4S$. Find the value of r. [3]

b An arithmetic progression has first term -24. The nth term is -13.8 and the $(2n)$th term is -3. Find the value of n. [4]

Examination style question